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Fully group graded algebras and a theorem of Fong

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Abstract

It is proved that any algebra fully graded by a finite group over a complete discrete valuation ring with an algebraically closed residue field of characteristic a prime p is Morita equivalent to an embedded graded subalgebra which is a crossed product; and an explicit way to get a decomposition of unity with a bounded length is shown. When the finite group is p-solvable, a theorem of Fong's type for fully graded algebras is obtained.

Keywords: Fully group graded algebra; Crossed product; Decomposition of unity; Primitive idempotents; Finite p-solvable group

1. Introduction

1.1. Let $\mathcal{O}$ be a complete discrete valuation ring with an algebraically closed residue field k of characteristic p where p is a prime; it is allowed that $\mathcal{O} = k$. Any $\mathcal{O}$-algebra A in this paper is unitary and $\mathcal{O}$-free of finite $\mathcal{O}$-rank; but note that a subalgebra B of A is not necessarily unitary and the unity 1_B of B is an idempotent of A, and B is called an *embedded subalgebra* of A if $B = 1_B A 1_B$. Let G be a finite group.

P. Fong in [4] showed that if G is p-solvable, then every projective indecomposable $\mathcal{O}G$-module is isomorphic to an induced module from a module of a Hall p'-subgroup. Conversely, as illustrated in [2, Example 13], the induced module of an indecomposable module of a Hall p'-subgroup is not necessarily indecomposable; a natural question emerges: what is

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about the induced modules? Reference [7] gave results in a character version of the Fong's type for π-separable groups. And [5] exhibited a G-algebra approach of the questions. As shown in Ref. [2], it is more interesting to consider the questions of the Fong's type for group graded algebras; and [2] gave a complete answer for crossed products, which covers the G-algebra approach of the questions. Later, [3] gave an answer for general graded algebras which are not necessarily fully graded.

In this paper we are concerned about the situation of fully graded algebras. The study of the question turns out a result of general interesting on group graded algebras: any fully G-graded algebra is Morita equivalent to an embedded graded subalgebra which is a crossed product of G; and the decomposition of unity of the fully graded algebra can be obtained explicitly by means of such subalgebras. This fact allows to translate to fully graded algebras any result on crossed products in terms of Morita equivalence classes. Consequently, the result of [2] is translated to the fully graded case.

1.2. Recall that a *point* α on an $\mathcal{O}$-algebra A means an A^*-conjugacy class of primitive idempotents on A, where A^* denotes the multiplicative group consisting of the invertible elements of A; and $\mathcal{P}(A)$ denotes the set of the points on A. For any non-zero idempotent e there is an orthogonal set $I_e \subset \bigcup_{\alpha\in\mathcal{P}(A)} \alpha$ such that $e = \sum_{i\in I_e} i$; and the number $m_\alpha^e = |I_e \cap \alpha|$ is independent of the choice of the orthogonal set I_e, and is called the *multiplicity* of the point α at e. If $m_\alpha^e = m_\beta^e$ for any $\alpha, \beta \in \mathcal{P}(A)$, we call e an *isotypic idempotent* of A; at that case, the embedded subalgebra $B = eAe$ is Morita equivalent to A in a natural way, see (2.2.2) below. In particular, if $m_\alpha^e = 1$ for any $\alpha \in \mathcal{P}(A)$, we call e a *basic idempotent* of A.

An $\mathcal{O}$-algebra A is said to be G-*graded* if A has $\mathcal{O}$-submodules A_x indexed by $x \in G$ such that $A = \bigoplus_{x\in G} A_x$ and $A_x A_y \subseteq A_{xy}$ for all $x, y \in G$. For $S \subseteq G$ the submodule $A_S = \bigoplus_{s\in S} A_s$ is called the S-*component* of A; in particular, A_x for $x \in G$ is named the x-component of A. Observe that the 1-component A_1 must be an unitary subalgebra of A, cf. [6]. A G-graded algebra A is said to be *fully* G-*graded* if $A_x A_y = A_{xy}$ for all $x, y \in G$; and A is said to be a *crossed product* of G over A_1 if for each $x \in G$ there is an $\hat{x} \in A_x \cap A^*$ such that $A_x = A_1\hat{x}$.

1.3. Theorem. *Let A be a G-graded $\mathcal{O}$-algebra and e_1 be an isotypic idempotent of the 1-component A_1. Then $B = e_1 A e_1$ is a graded subalgebra of A which is Morita equivalent to A, and A is a fully G-graded if and only if B is a crossed product of G over B_1.*

1.4. Let A be a fully G-graded algebra. For any $x \in G$, inasmuch as $A_x A_{x^{-1}} = A_1$, there are $a_x^{(1)}, \dots, a_x^{(n)} \in A_x$ and $b_{x^{-1}}^{(1)}, \dots, b_{x^{-1}}^{(n)} \in A_{x^{-1}}$ such that $1 = \sum_{i=1}^n a_x^{(i)} b_{x^{-1}}^{(i)}$, which is called a *decomposition of unity* at x, and we denominate n the *length* of the decomposition. In A_1 we consider the multiplicity m_α^1 of any $\alpha \in \mathcal{P}(A_1)$ at the unity element 1, and denote $m_{\max} = \max\{m_\alpha^1 \mid \alpha \in \mathcal{P}(A_1)\}$ and $m_{\min} = \min\{m_\alpha^1 \mid \alpha \in \mathcal{P}(A_1)\}$, and set

$$d = \lceil m_{\max}/m_{\min} \rceil$$

which denotes the least integer not less than $m_{\max}/m_{\min}$. As a corollary, for any $x \in G$ we show a way to find a decomposition of unity at x of length bounded by d.

1.5. Theorem. *With notation as above, at any $x \in G$ there is a decomposition of unity of length at most d.*

1.6. Remark. The above boundary d for the length of decompositions of unity is strict in some sense; for example, let $G = \{1, x\}$ be a group of order 2, and the matrix algebra $A = \mathrm{M}_4(k) = A_1 \oplus A_2$ where

$$A_1 = \left\{ \begin{pmatrix} a_{11} & & & \\ & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{pmatrix} \middle| \, a_{ij} \in k \right\}$$

and

$$A_x = \left\{ \begin{pmatrix} & a_{12} & a_{13} & a_{14} \\ a_{21} & & & \\ a_{31} & & & \\ a_{41} & & & \end{pmatrix} \middle| \, a_{ij} \in k \right\};$$

then it is easy to verify that A is a fully G-graded k-algebra, and the length of decomposition of unity at x is at least 3, since for any $a, b \in A_x$ the bottom right 3×3 block of ab has rank at most 1.

1.7. Let $A = \bigoplus_{x \in G} A_x$ be a fully G-graded $\mathcal{O}$-algebra. Recall from [2] that the *divisor monoid* $\mathcal{D}(A)$ of A is the free additive monoid generated by the set $\mathcal{P}(A)$, and $\mathcal{D}(A)$ is isomorphic to the positive part of the Grothendieck group of A. Let K and H be any subgroups of G with $K \subseteq H$. Then the inclusion map $A_K \to A_H$ induces a monoid homomorphism

$$\iota^A_{K,H} : \mathcal{D}(A_K) \to \mathcal{D}(A_H) \tag{1.7.1}$$

which sends any point $\beta \in \mathcal{P}(A_K)$ to $\sum_{\alpha \in \mathcal{P}(A_H)} m^j_\alpha \cdot \alpha \in \mathcal{D}(A_H)$, where $j \in \beta$; see 1.2 above or cf. [2, Section 4]. In particular, setting $H_x = H \cap H^x$ for $x \in G$ where $H^x = x^{-1}Hx$, we have a monoid homomorphism

$$\iota^A_H = \bigoplus_{x \in G} \iota^A_{H_x,H} : \bigoplus_{x \in G} \mathcal{D}(A_{H_x}) \to \mathcal{D}(A_H). \tag{1.7.2}$$

Further, any $\alpha \in \mathcal{P}(A_{K^x})$ determines a minimal idempotent ideal $\mathfrak{m}_\alpha = A_{K^x} \alpha A_{K^x}$ of A_{K^x}, see (2.1.3) below. And $A_x \mathfrak{m}_\alpha A_{x^{-1}}$ is obviously an idempotent ideal of A_K; conversely, it is clear that $A_{x^{-1}}(A_x \mathfrak{m}_\alpha A_{x^{-1}})A_x = \mathfrak{m}_\alpha$; that is, $A_x \mathfrak{m}_\alpha A_{x^{-1}}$ is a minimal idempotent ideal of A_K, and determines a point of A_K, denoted by $\alpha^{x^{-1}}$. In this way we have a bijection $\mathcal{P}(A_{K^x}) \to \mathcal{P}(A_K)$, hence we get a monoid isomorphism

$$\kappa^A_{K,x} : \mathcal{D}(A_{K^x}) \to \mathcal{D}(A_K). \tag{1.7.3}$$

Then we have a monoid homomorphism

$$\kappa^A_H = \bigoplus_{x \in G} \iota^A_{H_{x^{-1}},H} \circ \kappa^A_{H_{x^{-1}},x} : \bigoplus_{x \in G} \mathcal{D}(A_{H_x}) \to \mathcal{D}(A_H). \tag{1.7.4}$$

Once these are settle down, by Theorem 1.3, the main result of [2] can be translated:

1.8. Theorem. *Let G be a finite p-solvable group and $A = \bigoplus_{x\in G} A_x$ be a fully G-graded $\mathcal{O}$-algebra, and let H be a Hall p'-subgroup of G. Then the following is a coequalizer diagram*:

$$\bigoplus_{x\in G} \mathcal{D}(A_{H_x}) \underset{\kappa_H^A}{\overset{\iota_H^A}{\rightrightarrows}} \mathcal{D}(A_H) \xrightarrow{\iota_{H,G}^A} \mathcal{D}(A_G). \tag{1.8.1}$$

1.9. In Subsections 2.1–2.3 we show necessary notation and preliminaries as preparations. After some lemmas, Theorem 1.3 follows in 2.7; and Theorem 1.5 will be proved in Subsection 2.8. At last, some explanations are made for 1.7.

2. Proofs

2.1. Let p, $\mathcal{O}$, k and G always be the same as in 1.1. Let A be an $\mathcal{O}$-algebra, and $a \in A$. Denote $\bar{A} = A/J(A)$ where $J(A)$ denotes the Jacobson radical of A. Noting that $1 + r \in A^*$ (i.e. $1 + r$ is invertible) for any $r \in J(A)$, we have the following:

(2.1.1) *$a \in A^*$ if and only if its residual image $\bar{a} \in \bar{A}^*$.*

Further, the residual image $\bar{\alpha}$ of a point $\alpha \in \mathcal{P}(A)$ is a point of $\bar{A}$, and $\mathcal{P}(A) \to \mathcal{P}(\bar{A})$, $\alpha \mapsto \bar{\alpha}$, is a bijection. The residue algebra $A/J(A)$ is semisimple, i.e. a direct product of simple k-algebras; each of the simple factors corresponds exactly one point $\alpha \in \mathcal{P}(A)$, and the multiplicity $m_\alpha = m_\alpha^1$ of α at the unity 1, called the multiplicity of α in A, is preserved in the residue algebra $A/J(A)$; i.e.

$$A/J(A) \cong \prod_{\alpha\in\mathcal{P}(A)} \mathrm{M}_{m_\alpha}(k), \tag{2.1.2}$$

where $\mathrm{M}_{m_\alpha}(k)$ denotes the matrix algebra over k of order m_α; cf. [8, Section 2]. Namely, each point α on A corresponds exactly to one irreducible A-module and m_α is the multiplicity of the irreducible module appearing in the regular A-module. For $\alpha \in \mathcal{P}(A)$, from (2.1.2) it is easy to see that $\mathfrak{m}_\alpha = A\alpha A = \{\sum aia' \mid i \in \alpha,\ a, a' \in A\}$ is a minimal idempotent ideal of A which corresponds to $\mathrm{M}_{m_\alpha}(k)$ in (2.1.2), and in this way we have

(2.1.3) *the $\mathfrak{m}_\alpha$'s for $\alpha \in \mathcal{P}(A)$ are exactly all minimal idempotent ideals of A.*

2.2. Let B be a subalgebra of A. For $\beta \in \mathcal{P}(B)$ take $j \in \beta$, then in A the multiplicity m_α^j for $\alpha \in \mathcal{P}(A)$ is independent of the choice of $j \in \beta$, hence we can denote it by m_α^β; please consult [8, Section 2] for detail. In this way, the inclusion map $B \to A$ always induces a monoid homomorphism

$$\mathcal{D}(B) \to \mathcal{D}(A), \qquad \beta \mapsto \sum_{\alpha\in\mathcal{P}(A)} m_\alpha^\beta \cdot \alpha. \tag{2.2.1}$$

We remark that the $\iota_{K,H}^A$ in (1.7.1) is just such a homomorphism.

Next, let e be an idempotent of the algebra A and assume $B = eAe$. For any $\alpha \in \mathcal{P}(A)$, if $m_\alpha^e \neq 0$ then $eAe \cap \alpha$ is a point on eAe and m_α^e is just the multiplicity of the point $eAe \cap \alpha$

in eAe; otherwise, $eAe \cap \alpha = \emptyset$. In other words, in that case the homomorphism (2.2.1) maps a point $\beta \in \mathcal{P}(B)$ to a point $\alpha \in \mathcal{P}(A)$ fulfilling that $\beta \subset \alpha$. And then, the homomorphism in (2.2.1) is injective in this case. For the above, please refer to [8]; and the following is known in [8, Proposition 2.7]:

(2.2.2) *The following three statements are equivalent*:
- (i) $eAe \cap \alpha \neq \emptyset$ *for all* $\alpha \in \mathcal{P}(A)$;
- (ii) A *is Morita equivalent to* eAe *by sending an* A*-module* M *to the* eAe*-module* eM;
- (iii) $AeA = A$.

If $m_\alpha^e = m_\beta^e = m \neq 0$ for all $\alpha, \beta \in \mathcal{P}(A)$, then we call e an *isotypic idempotent* of A of multiplicity m. Further, if the unity 1 of A is an isotypic idempotent of multiplicity m, then A is named to be *isotypic of multiplicity* m.

2.3. Let $A = \bigoplus_{x \in G} A_x$ be a G-graded $\mathcal{O}$-algebra. It is easy to see that:

(2.3.1) A *is fully graded if and only if* $A_x A_{x^{-1}} = A_1$ *for all* $x \in G$.

If $c \in A^*$, then multiplication by c produces a bijection $A \to A$, $a \mapsto ac$. From this fact it is easy to descry that:

(2.3.2) *If* $a_x \in A_x \cap A^*$*, then* $a_x^{-1} \in A_{x^{-1}}$ *and* $A_x = A_1 a_x$.

An ideal I of A is said to be *graded* if $I = \bigoplus_{x \in G} I_x$ with $I_x = A_x \cap I$; at that case, A/I is a graded algebra with x*-component* isomorphic to A_x/I_x. It is well known that $AJ(A_1)A$ is a graded ideal of A contained in $J(A)$; cf. [1, Lemma 7]. On the other hand, if e is an idempotent of A_1, then $eAe = \bigoplus_{x \in G} eA_x e$ is an embedded and graded subalgebra of A; moreover, we have:

2.4. Lemma. *If* A *is a* G*-graded algebra and* e *is an idempotent of* A_1 *such that* eA_1e *is Morita equivalent to* A_1*, then*

- (i) eAe *is Morita equivalent to* A;
- (ii) eAe *is fully graded if and only if* A *is fully graded.*

Proof. By assumption we have $A_1eA_1 = A_1$, see (2.2.2); so $AeA = A \cdot (A_1eA_1) \cdot A = AA_1A = A$, and by (2.2.2) again, eAe is Morita equivalent to A.

If A is fully graded, for any $x \in G$ we have $eA_xe \cdot eA_{x^{-1}}e = eA_x \cdot (A_1eA_1) \cdot A_{x^{-1}}e = eA_xA_1A_{x^{-1}}e = eA_1e$, therefore eAe is fully graded with x-component eA_xe.

Conversely assume that $eAe = \bigoplus_{x \in G} eA_xe$ is fully graded. By the assumption of the lemma, for any $x \in G$ we have

$$A_x = A_1A_xA_1 = A_1eA_1A_xA_1eA_1 = A_1e(A_1A_xA_1)eA_1 = A_1eA_xeA_1;$$

hence for any $x, y \in G$ we have

$$A_xA_y = A_1eA_xeA_1A_1eA_yeA_1 \supseteq A_1eA_xeeA_yeA_1 = A_1eA_{xy}eA_1 = A_{xy};$$

that is, A is fully graded. $\square$

2.5. Lemma. *If A is a fully G-graded $\mathcal{O}$-algebra and its* 1*-component A_1 is isotypic of multiplicity* 1*, then A is a crossed product of G over A_1.*

Proof. First we assume that A_1 is semisimple. Then $\mathcal{O} = k$ and A_1 is commutative and every point of A_1 contains exactly one primitive idempotent i; cf. (2.1.2). Instead of $\{i\} \in \mathcal{P}(A_1)$, we write $i \in \mathcal{P}(A_1)$ for convenience. So we have that $A_1 = \bigoplus_{i\in\mathcal{P}(A_1)} ki$, where ki is the 1-dimensional ideal of A_1 generated by i; and any ideal of A_1 is idempotent. So, for any $x \in G$, we gain that

$$A_x A_{x^{-1}} = A_1 = \bigoplus_{i\in\mathcal{P}(A_1)} ki = A_x\Bigg(\bigoplus_{i\in\mathcal{P}(A_1)} ki\Bigg)A_{x^{-1}}.$$

On one hand, let $i \in \mathcal{P}(A_1)$, if $A_x i A_{x^{-1}} = 0$, then $ki = A_1 i A_1 = A_{x^{-1}} A_x i A_{x^{-1}} A_x = 0$, which is a contradiction. Therefore, $A_x i A_{x^{-1}} \neq 0$ for all $i \in \mathcal{P}(A_1)$. On the other hand, if $i \neq i' \in \mathcal{P}(A_1)$ then $A_x i A_{x^{-1}} \cdot A_x i' A_{x^{-1}} = 0$. In virtue of (2.1.2) we get that

$$\bigoplus_{i\in\mathcal{P}(A_1)} A_x i A_{x^{-1}} = \bigoplus_{i\in\mathcal{P}(A_1)} ki$$

and $A_x i A_{x^{-1}} \neq 0$ for all $i \in \mathcal{P}(A_1)$. Namely, there exists a permutation ρ of $\mathcal{P}(A_1)$ such that

$$A_x i \cdot i A_{x^{-1}} = A_x i A_{x^{-1}} = k \cdot \rho(i), \quad \forall i \in \mathcal{P}(A_1).$$

Hence there are $a_{x,i} \in A_x i$ and $b_{i,x^{-1}} \in i A_{x^{-1}}$ such that $a_{x,i} b_{i,x^{-1}} = \lambda \cdot \rho(i)$ with $0 \neq \lambda \in k$; replacing $a_{x,i}$ by $\lambda^{-1} a_{x,i}$, we acquire $a_{x,i} \in A_x i$ and $b_{i,x^{-1}} \in i A_{x^{-1}}$ such that

$$a_{x,i} b_{i,x^{-1}} = \rho(i).$$

Let

$$a_x = \sum_{i\in\mathcal{P}(A_1)} a_{x,i} \quad \text{and} \quad b_{x^{-1}} = \sum_{i\in\mathcal{P}(A_1)} b_{i,x^{-1}}.$$

Noting that the idempotents of $\mathcal{P}(A_1)$ are orthogonal to each other, one can check that

$$a_x b_{x^{-1}} = \sum_{i\in\mathcal{P}(A_1)} \rho(i) = \sum_{i\in\mathcal{P}(A_1)} i = 1.$$

Since A is a finite-dimensional k-algebra, such a_x is an invertible element of A. Therefore, $A_x = A_1 a_x$, and $A = \bigoplus_{x\in G} A_1 a_x$ is a crossed product of G.

For the general case, taking $I = AJ(A_1)A$ which is a graded ideal contained in $J(A)$, we have the residue algebra $\bar{A} = A/I = \bigoplus_{x\in G} \bar{A}_x$ which is fully G-graded with semisimple 1-component $\bar{A}_1 = A_1/J(A_1)$ which is isotypic of multiplicity 1. Thus, for any $x \in G$, there exists $\bar{a}_x \in \bar{A}^* \cap \bar{A}_x$; by the surjection $A_x \to \bar{A}_x$ we can choose an inverse image a_x of $\bar{a}_x$ in A_x. By using of (2.1.1) we see that $a_x \in A^* \cap A_x$. Then $A_x = A_1 \cdot a_x$ for any $x \in G$ and A is a crossed product of G over A_1. $\square$

2.6. Corollary. *If A is a fully G-graded $\mathcal{O}$-algebra with isotypic* 1*-component A_1, then A is a crossed product of G over A_1.*

Proof. Let $\mathcal{P}(A_1) = \{\alpha_1, \ldots, \alpha_h\}$. By the assumption, in A_1 we obtain an orthogonal primitive decomposition of 1 as follows:

$$1 = e_{11} + \cdots + e_{1m} + \cdots + e_{h1} + \cdots + e_{hm} \tag{2.6.1}$$

such that $e_{st} \in \alpha_s$ for $s = 1, \ldots, h$ and $t = 1, \ldots, m$. Consequently,

$$e_t = e_{1t} + \cdots + e_{ht}, \quad t = 1, \ldots, m,$$

are basic idempotents of A_1; hence, for each $t = 1, \ldots, m$, the embedded subalgebra $e_t A e_t$ is fully G-graded too, see 2.4, and its 1-component $e_t A_1 e_t$ is isotypic of multiplicity 1. By the above lemma, for any $x \in G$ we have $a_{t,x} \in e_t A_x e_t$ and $b_{t,x^{-1}} \in e_t A_{x^{-1}} e_t$ fulfilling that $a_{t,x} b_{t,x^{-1}} = e_t = b_{t,x^{-1}} a_{t,x}$. Let

$$a_x = a_{1,x} + \cdots + a_{m,x}, \qquad b_{x^{-1}} = b_{1,x^{-1}} + \cdots + b_{m,x^{-1}}.$$

Since $a_{t,x} = e_t a_{t,x} e_t$ and $b_{t,x^{-1}} = e_t b_{t,x^{-1}} e_t$ for $t = 1, \ldots, m$, and $e_1, \ldots, e_m$ are orthogonal to each other, it is evident that

$$a_x b_{x^{-1}} = a_{1,x} b_{1,x^{-1}} + \cdots + a_{m,x} b_{m,x^{-1}} = e_1 + \cdots + e_m = 1.$$

We can get $b_{x^{-1}} a_x = 1$ in the same way. Namely, $a_x \in A_x \cap A^*$. As a result, $A = \bigoplus_{x \in G} A_1 a_x$ is a crossed product of G over A_1. $\square$

2.7.

Proof of Theorem 1.3. It follows from Lemma 2.4 and Corollary 2.6. $\square$

2.8.

Proof of Theorem 1.5. By the assumption, in A_1 we have a set I of orthogonal primitive idempotents such that $1 = \sum_{e \in I} e$ and $|\alpha \cap I| \leqslant m_{\max}$ for all $\alpha \in \mathcal{P}(A_1)$.

First, we can take $I_1 \subseteq I$ such that for any $\alpha \in \mathcal{P}(A_1)$ we have $|\alpha \cap I_1| = m_{\min}$. Let $I^{(1)} = I - I_1$ and denote

$$m_{\max}^{(1)} = \max_{\alpha \in \mathcal{P}(A_1)} \left|\alpha \cap I^{(1)}\right| = m_{\max} - m_{\min}.$$

If $m_{\max}^{(1)} > 0$, take $I_2 \subseteq I^{(1)}$ such that

$$|\alpha \cap I_2| = \begin{cases} m_{\min}, & \text{if } |\alpha \cap I^{(1)}| \geqslant m_{\min}, \\ |\alpha \cap I^{(1)}|, & \text{if } |\alpha \cap I^{(1)}| < m_{\min}, \end{cases} \quad \forall \alpha \in \mathcal{P}(A_1).$$

Let $I^{(2)} = I^{(1)} - I_2$; then either $I^{(2)} = \emptyset$ or

$$m_{\max}^{(2)} = \max_{\alpha \in \mathcal{P}(A_1)} \left|\alpha \cap I^{(2)}\right| = m_{\max} - 2m_{\min} > 0.$$

And we can repeat the process, up to getting a partition $I_1, \ldots, I_d$ of I such that for any I_t with $1 \leqslant t \leqslant d$ and for any $\alpha \in \mathcal{P}(A_1)$ we have $|\alpha \cap I_t| \leqslant m_{\min}$. So we can find J_t with $I_t \subseteq J_t \subseteq I$ fulfilling that

$$|\alpha \cap J_t| = m_{\min}, \quad \forall \alpha \in \mathcal{P}(A_1).$$

Let $f_t = \sum_{e \in I_t} e$ and $f'_t = \sum_{e \in J_t} e$; then $f_t f'_t = f_t = f'_t f_t$, and f'_t is an isotypic idempotent on A_1 of multiplicity $m_{\min}$; by Corollary 2.6, for any $x \in G$, we attain an $a'_{t,x} \in A_x$ and a $b'_{t,x^{-1}} \in A_{x^{-1}}$ such that $f'_t = a'_{t,x} b'_{t,x^{-1}}$. Putting $a_{t,x} = f_t a'_{t,x} \in A_x$ and $b_{t,x^{-1}} = b'_{t,x^{-1}} f_t \in A_{x^{-1}}$, we have $f_t = a_{t,x} b_{t,x^{-1}}$. So we have

$$1 = \sum_{e \in I} e = \sum_{t=1}^{d} \sum_{e \in I_t} e = \sum_{t=1}^{d} f_t = \sum_{t=1}^{d} a_{t,x} b_{t,x^{-1}},$$

which is a decomposition at $x \in G$ of the unity with length at most d. And the proof of Theorem 1.5 is complete. □

2.9. Let $A = \bigoplus_{x \in G} A_x$ be a fully G-graded $\mathcal{O}$-algebra and e_1 be a basic idempotent on its 1-component A_1; set $B = e_1 A e_1$. Then, by Theorem 1.3, B is a crossed product of G over B_1; so for any $x \in G$ we take $\hat{x} \in B_x \cap B^*$ such that $B = \bigoplus_{x \in G} B_1 \hat{x}$. Further, for any subgroup H of G, by Theorem 1.3 again, the H-component $B_H = e_1 A_H e_1$ is Morita equivalent to A_H. By (2.2.1) and (2.2.2) we have a natural isomorphism

$$\delta_H : \mathcal{D}(B_H) \xrightarrow{\cong} \mathcal{D}(A_H). \tag{2.9.1}$$

Let K be a subgroup of G. In $B = \bigoplus_{x \in G} B_1 \hat{x}$, each $\hat{x}$ induces by conjugation an automorphism of B, it maps B_{K^x} onto B_K, and sends a point $\beta \in \mathcal{P}(B_{K^x})$ to a point $\hat{x}\beta\hat{x}^{-1} = \{\hat{x} j \hat{x}^{-1} \mid j \in \beta\}$ on B_K, hence, it induces a monoid isomorphism

$$\mathcal{D}(B_{K^x}) \to \mathcal{D}(B_K), \qquad \beta \mapsto \hat{x}\beta\hat{x}^{-1}, \tag{2.9.2}$$

which is the conjugation map defined in [2]. In notation of 1.7, it is easy to see that

$$B_x \mathfrak{m}_\beta B_{x^{-1}} = B_x B_{K^x} \beta B_{K^x} B_{x^{-1}} = B_K \hat{x}\beta\hat{x}^{-1} B_K;$$

that is, the map (2.9.2) coincides with the isomorphism $\kappa^B_{K,x}$ defined in (1.7.3) for B. Thus every thing in [2] is translated into fully graded algebras; in particular, we have a commutative diagram of monoids:

$$\begin{tikzcd}
\bigoplus_{x \in G} \mathcal{D}(B_{H_x}) \arrow[r, shift left, "\iota^B_H"] \arrow[r, shift right, "\kappa^B_H"'] \arrow[d, "\bigoplus_{x \in G} \delta_{H_x}"'] & \mathcal{D}(B_H) \arrow[r, "\iota^B_{H,G}"] \arrow[d, "\delta_H"] & \mathcal{D}(B_G) \arrow[d, "\delta_G"] \\
\bigoplus_{x \in G} \mathcal{D}(A_{H_x}) \arrow[r, shift left, "\iota^A_H"] \arrow[r, shift right, "\kappa^A_H"'] & \mathcal{D}(A_H) \arrow[r, "\iota^A_{H,G}"] & \mathcal{D}(A_G)
\end{tikzcd} \tag{2.9.3}$$

where the top line is, by the main result of [2], a coequalizer diagram; hereby the bottom line is a coequalizer too. That is the statement of Theorem 1.8.

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